The ideal of a subset of $\mathbb{A}^n$

We now know how to start with an ideal and get an algebraic set. What about going the other direction.

Again, let k be an algebraically closed field, $R = k[x_1, \dots, x_n]$. Let $X \subseteq \mathbb{A}^n$ be any subset.

Def: The <u>ideal</u> of X, $I(X)$, is the set of polynomials in R that vanish on X. i.e.

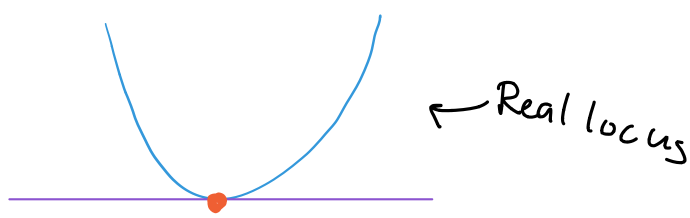
$$I(X) = \{ f \in R \mid f(P) = 0 \ \forall \ P \in X \}$$

Note: This is in fact an ideal! (check)

Naively, $I(-)$ looks like an "inverse" of $V(-)$. Not exactly.

Ex: Consider $X = \mathbb{Z} \subseteq \mathbb{A}^1_{\mathbb{C}}$. $\quad V(I(X)) = V(0) = \mathbb{A}^1$!

Ex: Consider $J = (y, y - x^2) \subseteq \mathbb{C}[x, y]$
$$= (y, x^2)$$



$V(J) = V(y) \cap V(x^2)$
$$= \{(0,0)\} \Rightarrow I(V(J)) = \{\text{polynomials s.t. } f(0,0) = 0\}$$
$$= (x, y) \neq J.$$

In general, how does J compare to $I(V(J))$?

If $f \in J$, then $f(P) = 0 \ \forall \ P \in V(J)$. $\Rightarrow f \in I(V(J))$

so $J \subseteq I(V(J))$. We can say more though!

Def: let R be a ring, $I \subseteq R$ an ideal. The radical of I is $\sqrt{I} := \{ a \in R \mid a^n \in I, \text{ some } n > 0 \}$

(Fulton calls this $\mathrm{Rad}(I)$)

(Note: $I \subseteq \sqrt{I}$)

I is a radical ideal if $\sqrt{I} = I$.

Lemma: $\sqrt{I}$ is an ideal — a radical ideal!

Pf: $a, b \in \sqrt{I} \Rightarrow a^n, b^m \in I$, some $n, m > 0$.

So $(a-b)^{n+m} = \sum_{\substack{i+j \\ = m+n}} \alpha_{ij} a^i b^j$, so $i \geq n$ or $j \geq m$

$\Rightarrow (a-b)^{n+m} \in I \Longrightarrow a - b \in \sqrt{I}$

If $c \in R$, then $(ca)^n = c^n a^n \in I \Rightarrow ca \in \sqrt{I} \Longrightarrow \sqrt{I}$ an ideal.

Consider $\sqrt{\sqrt{I}}$. If $a \in \sqrt{\sqrt{I}}$ then $a^n \in \sqrt{I}$, some $n > 0$, so

$a^{nm} \in I$, some $m > 0$, so $a \in \sqrt{I} \Rightarrow \sqrt{I}$ is radical. $\square$

Lemma: Let $R = k[x_1, \dots, x_n]$.

a.) If $J \subseteq R$ an ideal, then $\sqrt{J} \subseteq I(V(J))$.

b.) If $X \subseteq \mathbb{A}^n$, then $X \subseteq V(I(X))$.

Pf: a.) If $f \in \sqrt{J}$ then $f^n \in J$, some n. For $P \in V(J)$,
$f^n(P) = 0 \Rightarrow f(P) = 0 \Rightarrow f \in I(V(J))$.

b.) Let $P \in X$. Then if $f \in I(X)$, $f(P) = 0$. $\square$

Corollaries: (Exercise — check these!)

1.) $X \subseteq Y \Rightarrow I(X) \supseteq I(Y)$.

2.) $I(\emptyset) = k[x_1, \dots, x_n]$, $I(\mathbb{A}^n) = (0)$

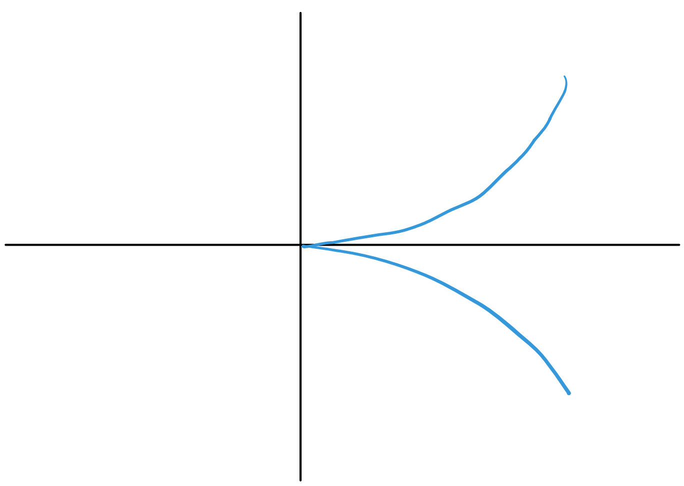
3.) $I(\{(a_1, \dots, a_n)\}) = (x_1 - a_1, \dots, x_n - a_n)$, for $a_i \in k$.

4.) $I(X)$ is a radical ideal.

5.) $V(I) = V(\sqrt{I})$ for $I \subseteq k[x_1, \dots, x_n]$ an ideal.

Examples:

1.) Cuspidal plane curve



Let $X = \{(t^2, t^3)\} \subseteq \mathbb{A}^2_{\mathbb{C}}$

Is this an algebraic set?

For a point $(x, y) \in X$, we have $x^3 - y^2 = 0$, so $X \subseteq V(x^3 - y^2)$.

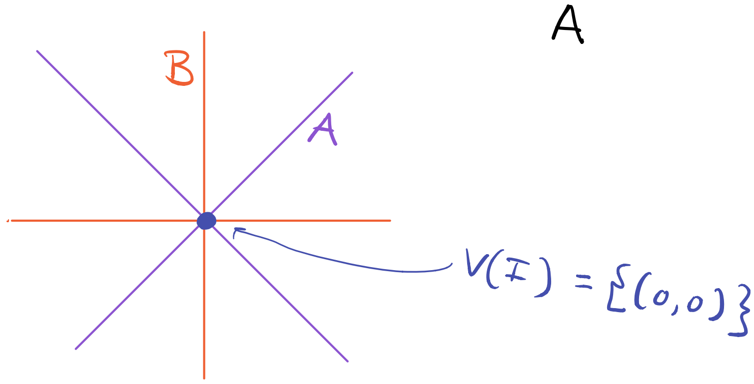
For the reverse inclusion, if $(a,b) \in V(x^3-y^2)$,

Choose t s.t. $t^2 = a$. Then $a^3 - b^2 = 0 \Rightarrow t^6 = b^2$. Thus, $\pm t^3 = b$, WLOG $t^3 = b$, so $(a,b) = (t^2, t^3) \in X$.

Tedious and ad hoc, but once we have more machinery it will be easier.

2.) Let $I = (x^2+y^2, x^2-y^2) \subseteq \mathbb{C}[x,y]$. What's $V(I)$?

Geometrically: $$V(I) = V(x^2+y^2) \cap V(x^2-y^2)$$
$$= \underbrace{\big(V(x+iy) \cup V(x-iy)\big)}_{A} \cap \underbrace{\big(V(x+y) \cup V(x-y)\big)}_{B}$$



More easily, $I = (x^2, y^2) \Rightarrow \sqrt{I} = (x,y) \Rightarrow V(I) = \{(0,0)\}$